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On the Numerical Solution of the Orr–Sommerfeld Problem: Asymptotic Initial Conditions for Shooting Methods

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A method, based on the use of certain asymptotic initial conditions together with the compound matrix method, is presented for the numerical solution of the Orr–Sommerfeld equation on infinite intervals. The asymptotic initial conditions require the numerical integration of a first-order nonlinear equation which is related to the "inviscid" solution and the use of a modified Liouville–Green approximation for the "viscous" solution. The method is applied to both the Blasius and the asymptotic suction boundary–layer profiles and it is shown that use of the asymptotic initial conditions requires a smaller interval of integration than the usual constant tail conditions. A discussion of a third-order eigenvalue problem is also given to illustrate how problems with a nonconstant tail can be treated by the compound matrix method.

1. INTRODUCTION

In the numerical solution of eigenvalue problems on infinite intervals, a common method of proceeding is to replace the infinite interval, $[0, \infty)$, say, by a finite one, $[0, z_\infty]$, say. The main problem then is to determine the appropriate boundary conditions to be imposed at $z = z_\infty$. In the "constant tail" case, z_∞ must be chosen sufficiently large so that the coefficients in the governing equation can be approximated for $z \geqslant z_\infty$ by their limiting values as $z \to \infty$. On this interval the limiting form of the governing equation can be solved exactly and the appropriate boundary conditions at $z = z_\infty$ then follow by requiring continuity of the solution and its derivatives at $z = z_\infty$.

When the governing equation contains a large parameter, however, it is often possible to obtain asymptotic approximations to the solutions with respect to the

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parameter. In this paper, therefore, we wish to show how this approach leads to certain "asymptotic boundary conditions" which can then be imposed at a point z_1, say, where z_1 can often be chosen significantly smaller than z_∞, thereby reducing the interval over which numerical integration is required.

The basic ideas involved in this asymptotic method were suggested by a study of the Orr–Sommerfeld equation which governs the stability of laminar boundary layers in the parallel flow approximation and this problem will be used therefore to describe the essential features of the method. Consider then the Orr–Sommerfeld equation

$$(i\alpha R)^{-1}(D^2-\alpha^2)^2\,\phi - \{(U-c)(D^2-\alpha^2)\phi - U''\phi\} = 0, \tag{1.1}$$

where $\phi(z)e^{i\alpha(x-ct)}$ is the disturbance stream function in the usual normal mode analysis, $U(z)$ is the basic velocity distribution, R is the Reynolds number based on the boundary-layer thickness, and $D = d/dz$. In this paper, we consider only the temporal stability problem in which the wave number α is taken to be real while the wave speed c is in general complex. The point z_c, where $U(z_c) - c = 0$ and $U'(z_c) \equiv U'_c \neq 0$, is called a simple turning point of (1.1) and it plays an important role in both the asymptotic and numerical treatment of the problem. The boundary conditions for this problem are

$$\phi(0) = \phi'(0) = 0 \tag{1.2a}$$

and

$$\phi(z), \phi'(z) \to 0 \qquad \text{as} \quad z \to \infty. \tag{1.2b}$$

For fixed values of α and R, Eqs. (1.1) and (1.2) thus define an eigenvalue problem with complex eigenvalue c.

In Section 2 we begin with a brief review of the constant tail conditions and then proceed to the derivation of the asymptotic initial conditions. For this purpose it is necessary to obtain asymptotic approximations to the bounded solutions of (1.1). One of these solutions is of "inviscid" type and leads to the integration of a first-order nonlinear equation; the other solution is of "viscous" type and leads to the use of the modified Liouville–Green approximations which are derived in the Appendix. In Section 3 we consider the application of these boundary conditions to the compound matrix method [1] and, in Section 4, we demonstrate the effectiveness of the asymptotic boundary conditions by computing certain unstable modes of the Blasius and the asymptotic suction boundary-layer profiles. These computations are intended to illustrate the effect of both moderate and large values of the Reynolds number. Finally, in Section 5, we give a brief discussion of a third-order eigenvalue problem which arises from perturbations about the Blasius boundary-layer profile. In particular, it is shown how the compound matrix method can be adapted to deal with problems, such as this one, in which the governing equation does not have a constant tail.

2. The Boundary Conditions at Infinity

2.1. *The Constant Tail Conditions*

To deal with the boundary conditions (1.2b), we first observe that as $z \to \infty$, $U(z) \to 1$ and $U''(z) \to 0$ for boundary-layer flows, and in this limit (1.1) becomes

$$(D^2 - \alpha^2)(D^2 - \beta^2)\phi = 0, \tag{2.1a}$$

where

$$\beta = [i\alpha R(1-c) + \alpha^2]^{1/2} \qquad \text{with} \qquad \mathrm{Re}(\beta) > 0. \tag{2.1b}$$

Thus, as $z \to \infty$ the bounded solutions of (1.1) have the asymptotic behavior

$$\phi_1(z) \sim \text{constant} \times e^{-\alpha z} \qquad \text{and} \qquad \phi_2(z) \sim \text{constant} \times e^{-\beta z}. \tag{2.2}$$

If we now choose z_∞ to be sufficiently large so that, for $z \geqslant z_\infty$, $U(z)$ and $U''(z)$ are numerically indistinguishable from their corresponding values at infinity, then ϕ_1, ϕ_2 and their derivatives can be treated as continuous at $z = z_\infty$. If, for convenience, we now let $\boldsymbol{\phi} = [\phi, \phi', \phi'', \phi''']^T$ then, on omitting the overall exponential factors, the constant tail initial conditions for $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ at $z = z_\infty$ are given by

$$\boldsymbol{\phi}_1 = [1, -\alpha, \alpha^2, -\alpha^3]^T \qquad \text{and} \qquad \boldsymbol{\phi}_2 = [1, -\beta, \beta^2, -\beta^3]^T. \tag{2.3a}$$

These initial conditions have customarily been used in connection with the method of shooting to obtain two linearly independent solutions of (1.1) by integrating from z_∞ to 0.

An equivalent form of these conditions, which are useful for some purposes, can be obtained by letting $\boldsymbol{\phi} = A\boldsymbol{\phi}_1 + B\boldsymbol{\phi}_2$. If the first two components of this equation are used to determine A and B, then the last two components lead to the conditions

$$\phi'' + (\beta + \alpha)\phi' + \alpha\beta\phi = 0$$

and (2.3b)

$$\phi''' - (\beta^2 + \alpha\beta + \alpha^2)\phi' - \alpha\beta(\beta + \alpha)\phi = 0$$

at $z = z_\infty$. In particular, when we integrate from 0 to z_∞, Eq. (2.3b) provides the necessary matching conditions at $z = z_\infty$.

The constant tail conditions (2.3a), (2.3b) are equivalent to those given by Itoh [2], Mack [3] and Keller [4] for the Orr–Sommerfeld problem. It is also of interest to note that conditions of this type had been used earlier by Brown [5] and Mack [6] in their work on the stability of compressible boundary layers and they have recently been discussed in greater generality by Keller [4].

2.2. *The Asymptotic Initial Conditions*

Consider now the possibility of deriving asymptotic approximations to the solutions ϕ_1 and ϕ_2 for $z_1 \leqslant z < \infty$, where $|z_1 - z_c| \geqslant \delta$ for some positive δ. A more precise estimate of δ will be given later.

It can be seen from (2.2) that ϕ_1 is largely inviscid in character whereas ϕ_2, being dependent on the Reynolds number R, is a solution of viscous type. According to the usual asymptotic theory [7], therefore, the first approximation to ϕ_1 is of the form

$$\phi_1(z) = \phi_1^{(0)}(z) + O\{(\alpha R)^{-1}\} \qquad \text{for} \quad |z - z_c| \geqslant \delta, \tag{2.4}$$

where $\phi_1^{(0)}$ is the solution of Rayleigh's equation

$$(U-c)(D^2 - \alpha^2)\phi_1^{(0)} - U''\phi_1^{(0)} = 0, \tag{2.5}$$

which has the asymptotic behavior $\phi_1^{(0)}(z) \sim \text{constant} \times e^{-\alpha z}$ as $z \to \infty$. To derive asymptotic approximations to the initial conditions for ϕ_1 at $z = z_1$, we first let

$$f_1 = \phi_1^{(0)\prime}/\phi_1^{(0)}, \qquad f_2 = \phi_1^{(0)\prime\prime}/\phi_1^{(0)}, \qquad f_3 = \phi_1^{(0)\prime\prime\prime}/\phi_1^{(0)}. \tag{2.6}$$

It then follows that f_1 must satisfy the first-order nonlinear equation

$$f_1' + f_1^2 - \left(\alpha^2 + \frac{U''}{U-c}\right) = 0 \tag{2.7}$$

and the initial condition $f_1(z_\infty^*) = -\alpha$, where $z_\infty^* \geqslant z_\infty$. A simple calculation also shows that

$$f_2 = \alpha^2 + \frac{U''}{U-c} \qquad \text{and} \qquad f_3 = f_1 f_2 + \frac{U'''}{U-c} - \frac{U'U''}{(U-c)^2}. \tag{2.8}$$

Hence, if we fix the normalization of ϕ_1 by requiring that $\phi_1(z_1) = 1$ then we can approximate ϕ_1 and its derivatives at $z = z_1$ with an error of the order of $(\alpha R)^{-1}$ by

$$\boldsymbol{\phi}_1 = [1, f_1, f_2, f_3]^T, \tag{2.9}$$

where f_1 is obtained by integrating (2.7) from z_∞^* to z_1, and f_2 and f_3 are given by (2.8). Equation (2.9) then provides the required asymptotic initial conditions for $\boldsymbol{\phi}_1$ at $z = z_1$.

To derive the corresponding initial conditions for the viscous solution ϕ_2, it is necessary to obtain an asymptotic approximation to ϕ_2 which is uniformly valid in the infinite interval $[z_1, \infty)$. The usual Liouville–Green (or WKBJ) approximation to ϕ_2, which has been widely used in the study of the stability of *bounded* flows, is not adequate for this purpose since it does not remain uniformly valid as $z \to \infty$. Instead, as discussed in the Appendix, it is necessary to use the modified Liouville–Green approximation

$$\phi_2(z) = \text{constant} \times (U-c)^{-5/4} e^{-\beta\eta}\{1 - \beta^{-1}H_1(z) + O(\beta^{-2})\}, \tag{2.10}$$

where $\eta(z)$ and $H_1(z)$ are given by Eqs. (A12) and (A13), respectively. For convenience, we have fixed the normalization of η and H_1 by requiring that $\eta(z_1)=0$ and $H_1(z_1)=0$. Clearly then we can approximate ϕ_2 and its derivatives at $z=z_1$, to within a common multiplicative factor, by

$$\begin{aligned}
\phi_2 &= 1,\\
\phi_2' &= -\beta\{\eta' + \tfrac{5}{4}g\beta^{-1} + O(\beta^{-2})\},\\
\phi_2'' &= \beta^2\eta'\{\eta' + 2g\beta^{-1} + O(\beta^{-2})\},\\
\phi_2''' &= -\beta^3\eta'^2\{\eta' + \tfrac{9}{4}g\beta^{-1} + O(\beta^{-2})\},
\end{aligned} \tag{2.11}$$

where

$$\eta' = \left(\frac{U-c}{1-c}\right)^{1/2} \quad \text{and} \quad g = \frac{U'}{U-c} \tag{2.12}$$

are to be evaluated at $z=z_1$. Equations (2.11) thus provide the required asymptotic initial conditions for $\boldsymbol{\phi}_2$ at $z=z_1$.

It can easily be seen that (2.9) and (2.11) reduce to the corresponding constant tail conditions as $z_1 \to z_\infty$ or ∞. Unlike (2.3a), (2.3b), however, the asymptotic approximations (2.9) and (2.11) remain valid even when U and U'' cannot be approximated by their values at infinity.

The accuracy of these parameter expansions clearly depends on the largeness of $|\beta|$. In dealing with problems for which $|\beta|$ is not large, however, the coordinate expansions suggested recently by Monkewitz and Monkewitz [8] would appear to offer a promising alternative.

3. The Compound Matrix Method

3.1. *Computation of the Eigenvalue*

For numerical purpose it is convenient to write the Orr–Sommerfeld equation as a system of first-order equations. As before, if we let $\boldsymbol{\phi} = [\phi, \phi', \phi'', \phi''']^T$ then Eq. (1.1) becomes

$$\boldsymbol{\phi}' = \mathbf{A}\boldsymbol{\phi}, \qquad \text{where} \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a_1 & 0 & a_3 & 0 \end{bmatrix} \tag{3.1}$$

and

$$\begin{aligned}
a_1 &= -\{\alpha^4 + i\alpha R[\alpha^2(U-c) + U'']\},\\
a_3 &= 2\alpha^2 + i\alpha R(U-c).
\end{aligned} \tag{3.2}$$

In the usual application of the shooting method, the solutions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are obtained separately by integrating (3.1) from either z_∞ or z_1 to $z=0$. For large values of the

Reynolds number, however, the Orr–Sommerfeld equation is inherently unstable and it is well known that $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ can quickly become numerically dependent. Several methods have been suggested to overcome this difficulty, including the method of orthonormalization [9–11], the Riccati method [12, 13] and the compound matrix method [1]. The asymptotic initial conditions (2.9) and (2.11) can be used in conjunction with each of these methods but we have chosen to use the compound matrix method for illustrative purposes throughout this paper.

The compound matrix method [1] is based on considering the 2×2 minors of the solution matrix $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2]$. These minors are given by

$$\begin{aligned} y_1 &= \phi_1\phi_2' - \phi_1'\phi_2, & y_4 &= \phi_1'\phi_2'' - \phi_1''\phi_2', \\ y_2 &= \phi_1\phi_2'' - \phi_1''\phi_2, & y_5 &= \phi_1'\phi_2''' - \phi_1'''\phi_2', \\ y_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, & y_6 &= \phi_1''\phi_2''' - \phi_1'''\phi_2'', \end{aligned} \tag{3.3}$$

and they satisfy the quadratic identity

$$y_1y_6 - y_2y_5 + y_3y_4 = 0. \tag{3.4}$$

On differentiating Eqs. (3.3) and using Eq. (1.1), it is easy to show that $y_1,..., y_6$ must satisfy the equations

$$\begin{aligned} y_1' &= y_2, & y_4' &= y_5, \\ y_2' &= y_3 + y_4, & y_5' &= -a_1y_1 + a_3y_4 + y_6, \\ y_3' &= a_3y_2 + y_5, & y_6' &= -a_1y_2. \end{aligned} \tag{3.5}$$

Consider now the constant tail initial conditions for $y_1,..., y_6$ at $z = z_\infty$. On substituting (2.3a) into (3.3), we obtain

$$\begin{aligned} y_1 &= \beta - \alpha, & y_4 &= \alpha\beta^2 - \alpha^2\beta, \\ y_2 &= -\beta^2 + \alpha^2, & y_5 &= -\alpha\beta^3 + \alpha^3\beta, \\ y_3 &= \beta^3 - \alpha^3, & y_6 &= \alpha^2\beta^3 - \alpha^3\beta^2, \end{aligned} \tag{3.6}$$

at $z = z_\infty$. Similarly, on substituting (2.9) and (2.11) into (3.3), we obtain the asymptotic initial conditions

$$\begin{aligned} y_1 &= \beta\{\eta' + (\tfrac{5}{4}g + f_1)\beta^{-1} + O(\beta^{-2})\}, \\ y_2 &= -\beta^2\eta'\{\eta' + 2g\beta^{-1} + O(\beta^{-2})\}, \\ y_3 &= \beta^3\eta'^2\{\eta' + \tfrac{9}{4}g\beta^{-1} + O(\beta^{-2})\}, \\ y_4 &= -\beta^2\eta'\{\eta' f_1 + (2f_1g + f_2)\beta^{-1} + O(\beta^{-2})\}, \\ y_5 &= \beta^3\eta'^2\{\eta' f_1 + \tfrac{9}{4}f_1g\beta^{-1} + O(\beta^{-2})\}, \\ y_6 &= \beta^3\eta'^2\{\eta' f_2 + (\tfrac{9}{4}f_2g + f_3)\beta^{-1} + O(\beta^{-2})\}, \end{aligned} \tag{3.7}$$

where η' and g, together with f_1, f_2 and f_3, are to be evaluated at $z = z_1$. The boundary conditions (1.2a) then lead to the eigenvalue relation

$$\begin{bmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{bmatrix} = 0 \qquad \text{or} \qquad y_1(0) = 0. \tag{3.8}$$

The solution of the Orr–Sommerfeld problem by the compound matrix method thus involves the integration of Eqs. (3.5), subject to the initial conditions given by either (3.6) or (3.7). An iterative procedure must then be used to vary the eigenvalue parameter c (say) until the eigenvalue relation (3.8) is satisfied.

3.2. *Computation of the Eigenfunctions*

Once the required eigenvalue has been obtained by the method just described, we can then proceed to the determination of the corresponding eigenfunction ϕ. In [1] it was shown that ϕ must satisfy the equations

$$y_1\phi'' - y_2\phi' + y_4\phi = 0, \tag{3.9}$$

$$y_1\phi''' - y_3\phi' + y_5\phi = 0, \tag{3.10}$$

$$y_2\phi''' - y_3\phi'' + y_6\phi = 0, \tag{3.11}$$

$$y_4\phi''' - y_5\phi'' + y_6\phi' = 0. \tag{3.12}$$

We now wish to show, however, that only Eq. (3.9) can be used to obtain ϕ by a forward integration from $z = 0$.

Consider first the constant tail initial conditions for which $y_1,..., y_6$ are known on the interval $[0, z_\infty]$. For $z > z_\infty$, Eqs. (3.9)–(3.12) have the asymptotic forms

$$\phi'' + (\beta + \alpha)\phi' + \alpha\beta\phi = 0, \tag{3.13}$$

$$\phi''' - (\beta^2 + \alpha\beta + \alpha^2)\phi' - \alpha\beta(\beta + \alpha)\phi = 0, \tag{3.14}$$

$$(\beta + \alpha)\phi''' + (\beta^2 + \alpha\beta + \alpha^2)\phi'' - \alpha^2\beta^2\phi = 0, \tag{3.15}$$

$$\phi''' + (\beta + \alpha)\phi'' + \alpha\beta\phi' = 0. \tag{3.16}$$

The roots of the characteristic equations associated with (3.13)–(3.16) are given by $(-\alpha, -\beta)$, $(-\alpha, -\beta, \beta + \alpha)$, $(-\alpha, -\beta, \alpha\beta/(\beta + \alpha))$, and $(-\alpha, -\beta, 0)$, respectively. Thus, as $z \to \infty$, any solution of Eq. (3.9) will automatically satisfy the boundary conditions (1.2b). Equations (3.10)–(3.12), however, admit solutions which do not decay to zero as $z \to \infty$. This difficulty is particularly severe in the case of Eq. (3.10) because the inevitable presence of some multiple of the solution $e^{(\beta+\alpha)z}$ will render a forward integration inherently unstable. In principle, therefore, only Eq. (3.9) can be used to obtain the eigenfunction ϕ.

It should be noted, of course, that Eq. (3.9) is singular at $z = 0$ and hence it is not possible to initiate the integration from the origin. This minor difficulty can easily be overcome, however, by integrating the Orr–Sommerfeld equation itself one step forward from 0 to h (say). For this purpose we note that if we fix the normalization of ϕ such that $\phi''(0) = 1$ then from (3.11) we have $\phi'''(0) = y_3(0)/y_2(0)$; alternatively, by using (3.12) or the quadratic identity (3.4), we have the equivalent condition $\phi'''(0) = y_5(0)/y_4(0)$. Thus, the initial conditions for $\boldsymbol{\phi}$ at $z = 0$ are completely specified. The values of $\boldsymbol{\phi}(h)$ can then be used as the initial conditions for Eq. (3.9). The solution of Eq. (3.9) obtained in this way clearly satisfies the boundary conditions at $z = 0$ and ∞. Moreover, by using an argument identical to the one given in [1], we can also show that ϕ is necessarily a solution of the Orr–Sommerfield equation and it is therefore the required eigenfunction.

In the case of the asymptotic initial conditions, $y_1, \ldots, y_6$ are known on the interval $|0, z_1|$ and on this interval the eigenfunction can be found by the method described above. For $z > z_1$, the integration of Eq. (3.9) can then be continued with y_1, y_2, and y_4 replaced by their uniform asymptotic approximations (3.7).

4. Numerical Examples

4.1. *The Blasius Boundary-Layer Profile*

To assess the effectiveness of the asymptotic initial conditions and the compound matrix method, we consider first the Orr–Sommerfeld problem for the Blasius boundary-layer profile. Thus we let $U(z) = F'(z)$, where $F(z)$ is the Blasius function defined by

$$F''' + \tfrac{1}{2}FF'' = 0 \tag{4.1a}$$

with

$$F(0) = F'(0) = 0 \qquad \text{and} \qquad F'(z) \to 1 \text{ as } z \to \infty. \tag{4.1b}$$

Following previous work on this problem we let $\alpha = 0.179$ and $R = 580$, and then consider the unstable mode for which Grosch and Orszag [14] have obtained the "exact" eigenvalue

$$c = 0.36412286 + i0.00795972. \tag{4.2}$$

For these values of the parameters, $|\beta|^2 \cong 66$ and such a small value of $|\beta|^2$ shows that this problem constitutes a severe test case for the accuracy of the asymptotic initial conditions. It should also be noted that the basic velocity profile for this problem tends to its free-stream value quite rapidly and this behavior is clearly advantageous when using the constant tail initial conditions.

The results obtained by applying both the constant tail and the asymptotic initial conditions are presented in Table I. All of the calculations were made on a CDC-6600 computer using a Runge–Kutta–Gill procedure with constant stepsize and single-precision arithmetic. In cases where the constant tail conditions were used, the system of equations (3.5) was integrated from z_∞ to 0 using a stepsize of 0.005. An

TABLE I

A Comparison of the Eigenvalue c Obtained by Using the Constant Tail and the Asymptotic Initial Conditions for the Blasius Boundary Layer with $\alpha = 0.179$ and $R = 580$[a]

	Constant Tail			Asymptotic		
z_∞ or z_1	c_r	c_i	k	c_r	c_i	k
10	.36412286	.00795972	8	.36412290	.00795974	8
9	.-------9	.-------1	7	.-------1	.-------5	7
8	.-----436	.------17	6	.------89	.------64	7
7	.----6069	.----4571	4	.------17	.-----762	6
6	.---69837	.---74582	3	.----1514	.----3860	5
5	.--909075	.--595536	2	.---08436	.---85430	4
4	.-8607840	−.00421576	*	.---05850	.---77544	3
3	.40733290	−.03551736	*	.---22744	.--830916	3
2	.39501348	−.06570021	*	.---40740	.-1009819	2

[a] The relative errors corresponding to different choices of z_∞ and z_1 are of the order of 10^{-k}.

iteration was then performed on the eigenvalue c until the eigenvalue relation (3.8) was satisfied. The behavior of y_1 near the origin is shown in Fig. 1. The values of c given in Table I should be compared with the "exact" value (4.2). It can readily be seen that the accuracy of the computed eigenvalue, though excellent when $z_\infty = 10$, deteriorates rapidly as z_∞ takes on succesively smaller values. This is entirely to be expected because the validity of the constant tail conditions depends crucially on U and U'' satisfying the free-stream conditions at $z = z_\infty$.

When using the asymptotic initial conditions, it is first necessary to integrate Eq. (2.7) from z_∞^* to z_1 to obtain $f_1(z_1)$. For the present calculations, we let $z_\infty^* = 10$ as the errors in approximating U and U'' at $z = 10$ by 1 and 0, respectively, are then of the order of 10^{-9}. We also found that it is sufficient to integrate (2.7) using a stepsize of 0.05. The behavior of f_1 for $2 \leqslant z \leqslant 10$ is shown in Fig. 2. We observe that f_1 begins to deviate significantly from its initial (constant tail) value for $z < 7$. This, of course, clearly indicates that U and U'' can no longer be approximated by their free-stream values and the behavior of f_1 is consistent therefore with the observed loss of accuracy in the eigenvalue computations using the constant tail initial conditions. Nevertheless, it should be emphasized that f_1 remains a valid approximation to ϕ_1'/ϕ_1 for $|z - z_c| \geqslant \delta > 0$. A more precise estimate for δ can be obtained from the asymptotic theory of the Orr–Sommerfeld equation. According to that theory, the behavior of ϕ_1 is essentially inviscid provided $|(z - z_c)\,(\alpha R U_c')^{1/3}| \gtrsim 10$. [There is also a certain domain in the complex z-plane in which ϕ_1 exhibits viscous behavior but this is not relevant to the present discussions.] For the present values of the prameters, this criterion gives $\delta \cong 3.1$ and, since $|z_c| \cong 1.2$, we see that the least value of z_1 for which the asymptotic initial conditions can be used with confidence is about 4.3.

Once the values of f_1 and hence f_2 and f_3 are known at $z = z_1$, Eqs. (3.5) can be

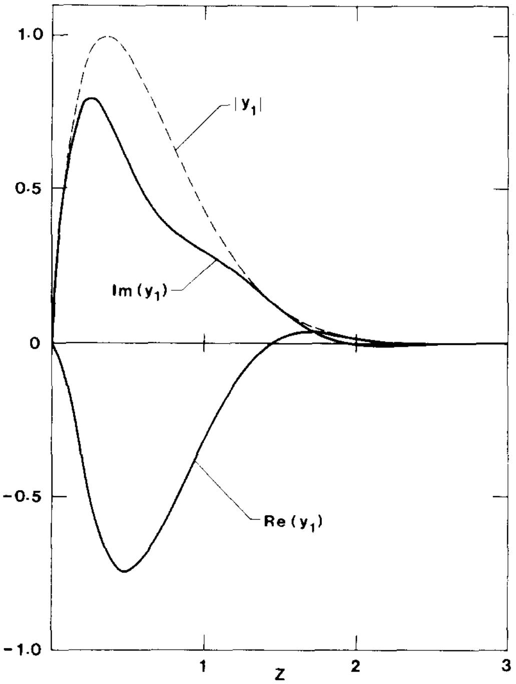

FIG. 1. The behavior of y_1 for the Blasius boundary-layer profile with $\alpha = 0.179$, $R = 580$, $c = 0.36412286 + i0.00795972$, and the maximum of $|y_1|$ normalized to unity.

integrated from z_1 to 0 subject to the asymptotic initial conditions (3.7). The values of c given in Table I for different choices of z_1 were all computed using a stepsize of 0.005. By comparing these results to those obtained using the constant tail conditions, it is clear that the eigenvalue can be determined to a certain prescribed accuracy by integrating (3.5) over a smaller interval if the asymptotic initial conditions are used. Since the additional work required for the numerical solution of (2.7) is relatively insignificant, a considerable saving in computational labor is thus possible.

It is also of some interest to compare the asymptotic approximations (3.7) with the values of $y_1,..., y_6$ obtained by the numerical solution of (3.5). For this purpose we normalize $y_2,..., y_6$ with respect to y_1 (for $z > 0$) by defining the ratios

$$q_2 = y_2/y_1, \qquad q_3 = y_3/y_1, \qquad q_4 = y_4/y_1, \qquad q_5 = y_5/y_1, \qquad q_6 = y_6/y_1. \quad (4.3)$$

The behavior of $q_2,..., q_6$ based on the asymptotic approximations (3.7) and the numerical solution of (3.5) are shown in Fig. 3 for $2 \leqslant z \leqslant 10$. It can readily be seen from these results that (3.7) provide, to within an overall multiplicative factor, excellent approximations to $y_1,..., y_6$ even for relatively small values of z. This, of course, is the basis for the success of the asymptotic initial conditions.

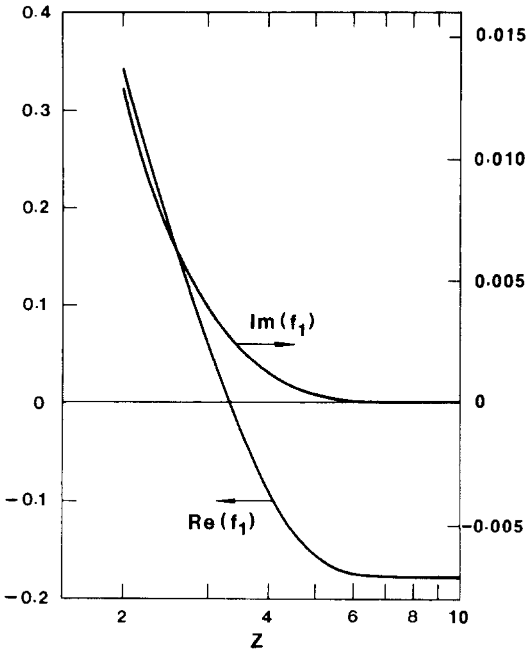

FIG. 2. The behavior of f_1 for the Blasius boundary-layer profile.

In the present calculations, we have used an exessively small integration stepsize in order to isolate the effect of the magnitudes of z_∞ and z_1 on the accuracy of the eigenvalues. When using the asymptotic initial conditions, it is sufficient for most purposes to integrate (3.5) from $z_1 \cong 5$ using a stepsize of 0.025 to obtain results correct to four significant figures for the given values of α and R.

We have also used the procedure described in Section 3.2 to compute the corresponding eigenfunction and the results obtained using both sets of initial conditions were found to be in excellent agreement with those given by Itoh [2].

4.2. *The Asymptotic Suction Boundary-Layer Profile*

As a second example, consider the asymptotic suction boundary-layer profile for which $U(z) = 1 - e^{-z}$. This is a flow for which instability is known to occur for large values of the Reynolds number. It is also a flow which tends to its free-stream value more slowly than the Blasius boundary layer. Indeed, to approximated U and U'' by their respective free-stream values with an error of the order of 10^{-7}, it is necessary to take $z \geqslant 16$. It can be expected therefore that the use of the asymptotic initial conditions will lead to a very substantial reduction in the length of the interval over which (3.5) must be integrated. To illustrate this point, we have computed the eigenvalue for the unstable mode with $\alpha = 0.14$ and $R = 10^5$ using both the constant tail and the asymptotic initial conditions, and the results, corresponding to various choices of z_∞ and z_1, are presented in Table II.

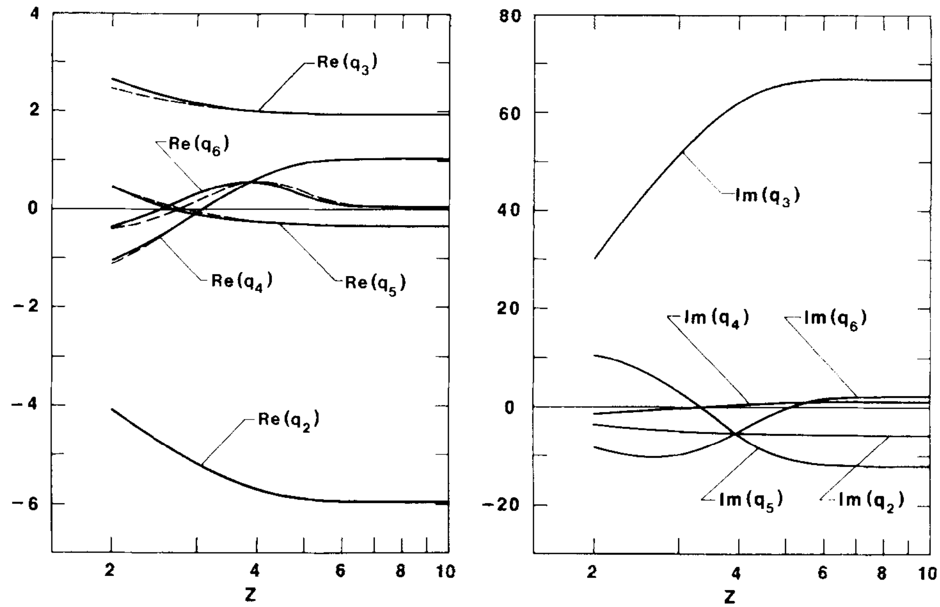

FIG. 3. The behavior of the real and the imaginary parts of the ratios $q_2,\ldots,q_6$ for the Blasius boundary-layer profile. Results based on the numerical solution of (3.5) and the asymptotic approximations (3.7) are shown in solid and dashed lines, respectively. Note that the scales of the figures for the real and the imaginary parts differ by a factor of 10.

TABLE II

A Comparison of the Eigenvalue c Obtained by Using the Constant Tail and the Asymptotic Initial Conditions for the Asymptotic Suction Boundary Layer with $\alpha = 0.14$ and $R = 10^5$[a]

	Constant Tail			Asymptotic		
z_∞ or z_1	c_r	c_i	k	c_r	c_i	k
16	.13240567	.00296407	6	.13240567	.00296407	6
12	.------73	.-------6	6	.--------	.--------	6
8	.----1727	.-----243	4	.--------	.--------	6
7	.----4732	.----5816	4	.-------6	.-------	6
6	.---55446	.----4272	3	.-------6	.-------6	6
5	.---93094	.---88592	3	.-------6	.-------4	6
4	.--420103	.---66437	2	.-------5	.-----397	6
3	.--804378	.--169051	2	.-------2	.-----373	6
2	.-4675156	−.00274370	*	.------55	.-----312	5
1	.-5549701	−.01579627	*	.------95	.-----639	5

[a] The relative errors corresponding to different choices of z_∞ and z_1 are of the order of 10^{-k}.

In the present calculations, we let $z_\infty^* = 16$ and it is again sufficient to integrate (2.7) using a stepsize of 0.05. On the other hand, for a fixed value of z_∞ or z_1 and regardless of which set of initial conditions were used, it was necessary to integrate (3.5) using a stepsize of 0.000625 to obtain convergence in the first seven digits of the eigenvalue. We found, however, that in order to determine an eigenvalue with a relative error of the order of 10^{-5} (say), approximately a 10-fold reduction in CPU time is possible if the asymptotic rather than the constant tail conditions are used. This observation is consistent with the results given in Table II and, incidentally, it shows that the computational labor required for integrating (2.7) is in general negligible. It should also be noted that if an eigenvalue with about four significant figures is required, it is entirely adequate to let $z_1 \cong 1$ and $h = 0.005$ for the present values of α and R. These rather dramatic results are, of course, a direct consequence not only of the asymptotic character of the approximations (3.7) but also of the largeness of $|\beta|^2$ ($\cong 1.2 \times 10^4$). Furthermore, as discussed in Section 4.1, ϕ_1 is essentially inviscid provided $|(z - z_c)(\alpha R U_c')^{1/3}| \gtrsim 10$ and, for the present values of the parameters, this criterion gives $\delta \cong 0.44$. Since $|z_c| \cong 0.14$, the least value of z_1 which can be used with confidence is about 0.58.

5. A Third-Order Eigenvalue Problem with a Nonconstant Tail

Thus far we have restricted our attention to the numerical solution of singular eigenvalue problems of the Orr–Sommerfeld type which have a constant tail as $z \to \infty$. In this section, we wish to show, by means of a simple example, how the basic ideas presented in this paper can be applied to eigenvalue problems with a nonconstant tail. For this purpose, we shall discuss briefly a problem involving a third-order equation which is of some fluid mechanical interest.

Consider then the equation

$$\phi''' + F\phi'' + \sigma F'\phi' + (1 - \sigma)F''\phi = 0, \tag{5.1}$$

where σ is a real parameter and ϕ satisfies the boundary conditions

$$\phi(0) = \phi'(0) = 0, \tag{5.2a}$$

and

$$\phi'(z) \to 0 \qquad \text{exponentially as } z \to \infty. \tag{5.2b}$$

In Eq. (5.1), $F(z)$ is again the Blasius function but, in keeping with previous work on this problem, we now take it to be the solution of the equation $F''' + FF'' = 0$ rather than Eq. (4.1a) but with the same boundary conditions (4.1b). The eigenvalue problem defined by (5.1) and (5.2) first arose in the study by Stewartson [15] of certain perturbation solutions in boundary-layer theory and since then it has been the subject of numerous investigations [16–18]. More recently, it has also been examined by Wilks and Bramley [19] in connection with the application of the Riccati method to odd-order differential systems, and Bramley [20] has further concluded that this problem provides an example for which neither the compound matrix method nor the method of orthonormalization is applicable. From the following discussion, however, it will be clear that Bramley's conclusions are erroneous and that, in fact, the compound matrix method provides a particularly simple method of treating the problem.

To apply the procedure described in the preceding sections, we now let $\boldsymbol{\phi} = [\phi, \phi', \phi'']^T$. If, as before, we let $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ denote two linearly independent solutions of (5.1) which satisfy the boundary condition (5.2b), then the 2×2 minors of the solution matrix are

$$y_1 = \phi_1 \phi_2' - \phi_1' \phi_2, \qquad y_2 = \phi_1 \phi_2'' - \phi_1'' \phi_2, \qquad y_3 = \phi_1' \phi_2'' - \phi_1'' \phi_2'. \tag{5.3}$$

On differentiating (5.3) and then eliminating the third derivatives by the use of (5.1), we have

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\sigma F' y_1 - F y_2 + y_3, \\ y_3' &= (1-\sigma) F'' y_1 - F y_3. \end{aligned} \tag{5.4}$$

The eigenvalue relation is then simply $y_1(0) = 0$, and the initial condition for $\mathbf{y} = [y_1, y_2, y_3]^T$ at some finite value of z follows directly from the corresponding conditions on $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ which we shall now derive.

For this purpose, we note that as $z \to \infty$, the asymptotic form of (5.1) is

$$\phi''' + \zeta \phi'' + \sigma \phi' = 0, \tag{5.5}$$

where $\zeta = z - a$ and $a = 1.216781$. Clearly then one of the two bounded solutions of (5.1), ϕ_1 (say), has the behavior $\phi_1 \sim$ constant. If we again choose z_∞ to be a point such that for $z \geqslant z_\infty$ the Blasius function and its derivatives are numerically indistinguishable from their respective free-stream values, then to within a constant multiplicative factor the initial condition for $\boldsymbol{\phi}_1$ at any $z \geqslant z_\infty$ is given by

$$\boldsymbol{\phi}_1 = [1, 0, 0]^T. \tag{5.6}$$

To obtain the initial condition for the second bounded solution $\boldsymbol{\phi}_2$, we shall first rewrite (5.5) in normal form. Thus, on substituting $\phi' = \exp(-\tfrac{1}{4}\zeta^2)\psi$ into (5.5), we obtain Weber's equation

$$\psi'' - [\tfrac{1}{4}\zeta^2 - (\sigma - \tfrac{1}{2})]\psi = 0. \tag{5.7}$$

It is useful to note that (5.7) has a simple turning point on the positive ζ-axis at

$\zeta_0 = 2(\sigma - \frac{1}{2})^{1/2}$. For fixed σ and $\zeta \gg \zeta_0$, the exponentially decaying solution of (5.7) has the asymptotic behavior

$$\psi \sim \zeta^{\sigma-1} \exp(-\tfrac{1}{4}\zeta^2), \tag{5.8}$$

where we have adopted the usual normalization for the principal solution of (5.7), and this gives

$$\phi_2' \sim \zeta^{\sigma-1} \exp(-\tfrac{1}{2}\zeta^2) \qquad \text{and} \qquad \phi_2'' \sim -\zeta^{\sigma} \exp(-\tfrac{1}{2}\zeta^2). \tag{5.9}$$

To approximate the initial condition for $\boldsymbol{\phi}_2$ at some finite value of $z = \hat{z}_\infty$ (say), we must require not only that $\hat{z}_\infty \geqslant z_\infty$ but also that $\hat{z}_\infty \gg z_0$, where $z_0 = \zeta_0 + a$. It is also convenient to normalize $\boldsymbol{\phi}_2$ so that $\phi_2(\hat{z}_\infty) = 0$ and then, on omitting an overall multiplicative factor, we have

$$\boldsymbol{\phi}_2 = [0, 1, -\hat{\zeta}_\infty]^T, \tag{5.10}$$

where $\hat{\zeta}_\infty = \hat{z}_\infty - a$. The corresponding initial condition for $\mathbf{y}$ at $z = \hat{z}_\infty$ follows directly from (5.6) and (5.10), and it is given by

$$\mathbf{y} = [1, -\hat{\zeta}_\infty, 0]^T. \tag{5.11}$$

From this discussion, however, it is evident that the choice of $\hat{z}_\infty$ for which (5.10) and (5.11) are valid depends critically on the value of σ. In the computation of the higher modes for which the values of σ become large, it is often necessary to take $\hat{z}_\infty \gg z_\infty$. To avoid the need to integrate (5.4) over an excessively long interval, it is of some practical importance to consider the possibility of formulating initial conditions for $\boldsymbol{\phi}_2$ and $\mathbf{y}$ which can be applied at $z = z_\infty$. The discussion in Section 2 suggests that we let $f = \phi_2''/\phi_2'$ and it then follows from (5.5) and (5.9) that f satisfies the *first-order* nonlinear equation

$$f' + \zeta f + f^2 + \sigma = 0 \qquad \text{with} \quad f(\hat{z}_\infty) = -\hat{\zeta}_\infty. \tag{5.12}$$

If we now normalize $\boldsymbol{\phi}_2$ so that $\phi_2(\hat{z}_\infty) = 0$ then we can approximate the initial condition for $\boldsymbol{\phi}_2$ to within a multiplicative constant by

$$\boldsymbol{\phi}_2 = [0, 1, f(z)]^T \qquad \text{at} \qquad z = z_\infty, \tag{5.13}$$

where $f(z_\infty)$ is obtained by integrating (5.12) from $\hat{z}_\infty$ to z_∞. The corresponding initial condition for $\mathbf{y}$ is therefore given by

$$\mathbf{y} = [1, f(z), 0]^T \qquad \text{at} \quad z = z_\infty. \tag{5.14}$$

When $z_\infty = \hat{z}_\infty$, it is clear that (5.13) and (5.14) are reducible to (5.10) and (5.11), respectively.

The numerical solution of this eigenvalue problem then consists of integrating (5.4) from either z_∞ or $\hat{z}_\infty$ to 0, together with a Newtonian iteration scheme to determine

σ. We have computed the values of σ for the first 20 modes, using both of the initial conditions (5.11) and (5.14), and the results thus obtained are in excellent agreement with those obtained previously [17, 19].

It is also of some interest to note that the initial conditions (5.6) together with (5.10) or (5.13) can be used to obtain two linearly independent solutions of (5.1) and thus, contrary to the observation of Bramley [20], the usual method of shooting, together with orthonormalization, is equally applicable to this problem. Moreover, we believe that further refinements in the various initial conditions are possible, but this would involve exploiting some of the special features of (5.1). We have not pursued this line of inquiry, however, since our primary purpose has simply been to illustrate how some of the basic ideas described in the earlier sections of this paper can be applied to singular eigenvalue problems on infinite intervals which are not of the Orr–Sommerfeld type.

APPENDIX: The Liouville–Green Approximations

The usual Liouville–Green (or WKBJ) approximations to the viscous solutions of the Orr–Sommerfeld equation were first derived by Heisenberg [21] and they have been used for a variety of purposes since then. In a second approximation to the bounded solution $\phi_2(z)$ we have [22]

$$\phi_2(z) = \text{constant} \times (U-c)^{-5/4} e^{-\lambda\xi}\{1 - \lambda^{-1} G_1(z) + O(\lambda^{-2})\}, \tag{A1}$$

where

$$\lambda = (i\alpha R U_c')^{1/2} \qquad \text{with} \quad \operatorname{Re}(\lambda) > 0, \tag{A2}$$

$$\xi(z) = \int_{z_c}^{z} \left(\frac{U-c}{U_c'}\right)^{1/2} dz, \tag{A3}$$

and

$$G_1(z) = \left(\frac{101}{48}\frac{U'}{U-c} + \frac{23}{24}\frac{U''}{U'}\right)\left(\frac{U_c'}{U-c}\right)^{1/2} - \int_{z_c}^{z} \left\{\frac{23}{24}\left(\frac{U'''}{U'} - \frac{U''^2}{U'^2}\right) - \frac{1}{2}\alpha^2\right\}\left(\frac{U_c'}{U-c}\right)^{1/2} dz. \tag{A4}$$

The lower limit of integration in Eq. (A3) is arbitrary but, without loss of generality, is usually chosen to be z_c as indicated. Similarly, $G_1(z)$ is only defined to within an arbitrary additive constant which, following Eagles [23], we have taken to be zero. For bounded flows with c real and $U(z)$ monotone increasing, Eq. (A1) provides a uniform approximation to $\phi_2(z)$ in the domain

$$-\tfrac{5}{4}\pi < \operatorname{ph} \xi(z) < \tfrac{3}{4}\pi \qquad \text{and} \qquad k \leqslant |z - z_c| \leqslant K, \tag{A5}$$

where $0 < k < K < \infty$. Furthermore, in this domain approximation (A1) is valid in the complete sense of Watson [24, p. 543]. For unbounded flows of the boundary layer type, however, it follows immediately from Eq. (A4) that

$$G_1(z) \sim \frac{1}{2}\alpha_2 \left(\frac{U_c'}{1-c}\right)^{1/2} z \qquad \text{as} \quad z \to +\infty \tag{A6}$$

and hence approximation (A1) does not remain uniformly valid as $z \to +\infty$.

In this Appendix, therefore, we wish to consider a modified form of the Liouville–Green approximation $\phi_2(z)$ which does remain uniformly valid as $z \to +\infty$. For this purpose we first observe that as $z \to +\infty$ the Orr–Sommerfeld equation becomes

$$(D^2 - \alpha^2)(D^2 - \beta^2)\phi = 0, \tag{A7}$$

where

$$\beta = [i\alpha R(1-c) + \alpha^2]^{1/2} \qquad \text{with} \quad \mathrm{Re}(\beta) > 0. \tag{A8}$$

Thus, as $z \to +\infty$, the bounded solutions of the Orr–Sommerfeld equation have the asymptotic behavior

$$\phi_1(z) \sim \text{constant} \times e^{-\alpha z} \qquad \text{and} \qquad \phi_2(z) \sim \text{constant} \times e^{-\beta z}. \tag{A9}$$

These approximations do not depend on the magnitude of β but when they are used for numerical purposes z_∞ must be chosen sufficiently large so that U and U'' are sensibly equal to 1 and 0, respectively.

In the analogous situation for second-order equations, Olver [25] has shown how the Liouville–Green approximation can be rendered uniform at infinity by a simple redefinition of the large parameter. This corresponds in the present case to choosing β rather than λ as the large parameter of the problem. Accordingly, we now rewrite the Orr–Sommerfeld equation in the form

$$(D^2 - \alpha^2)^2 \phi - (\beta^2 - \alpha^2) \left\{ \frac{U-c}{1-c} (D^2 - \alpha^2)\phi - \frac{U''}{1-c}\phi \right\} = 0. \tag{A10}$$

The Liouville–Green approximations to the viscous solutions of this equation can then be derived in the usual manner and, in a second approximation, we obtain

$$\phi_2(z) = \text{constant} \times (U-c)^{-5/4} e^{-\beta\eta} \{1 - \beta^{-1} H_1(z) + O(\beta^{-2})\}, \tag{A11}$$

where

$$\eta(z) = \int_{z_1}^{z} \left(\frac{U-c}{1-c}\right)^{1/2} dz \tag{A12}$$

and

$$H_1(z) = \left(\frac{101}{48}\frac{U'}{U-c} + \frac{23}{24}\frac{U''}{U'}\right)\left(\frac{1-c}{U-c}\right)^{1/2}$$
$$-\int_{z_1}^{z}\left\{\frac{23}{24}\left(\frac{U'''}{U'} - \frac{U''^2}{U'^2}\right) - \frac{1}{2}\alpha^2\right\}\left(\frac{1-c}{U-c}\right)^{1/2} dz$$
$$-\frac{1}{2}\alpha^2\eta + C. \tag{A13}$$

In Eq. (A12), the lower limit of integration is again arbitrary but, for numerical purposes, it is convenient to choose it as we have done so that $\eta(z_1) = 0$. Similarly, it is convenient to choose the arbitrary constant C in Eq. (A13) so that $H_1(z_1) = 0$ and this gives

$$C = -\left[\left(\frac{101}{48}\frac{U'}{U-c} + \frac{23}{24}\frac{U''}{U'}\right)\left(\frac{1-c}{U-c}\right)^{1/2}\right]_{z=z_1}. \tag{A14}$$

The essential feature of this modified Liouville–Green approximation is that $H_1(z) \sim$ constant as $z \to +\infty$ and approximation (A11) therefore remains uniformly valid at infinity.

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